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A note on Hoffman-type identities of graphs

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Abstract

An eigenvalue of a graph G is called main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. Hoffman [A.J. Hoffman, On the polynomial of a graph, Amer. Math. Monthly 70 (1963) 30–36] proved that G is a connected k-regular graph if and only if $n\prod_{i=2}^{t}(A-\lambda_i I)=\prod_{i=2}^{t}(k-\lambda_i)\cdot J$, where I is the unit matrix and J the all-one matrix and $\lambda_1=k,\lambda_2,\dots,\lambda_t$ are all distinct eigenvalues of adjacency matrix $A(G)$. In this note, some generalizations of Hoffman identity are presented by means of main eigenvalues.

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1. Introduction

Let G be a graph with vertex set $V=\{v_1,v_2,\dots,v_n\}$ and edge set E and adjacency $A=A(G)$ with eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$. We call $\lambda_1,\lambda_2,\dots,\lambda_n$ the eigenvalues of graph G. An eigenvalue of a graph G is called *main eigenvalue* if it has an eigenvector the sum of whose entries is not equal to zero. We denote by $m(G)$

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the number of main eigenvalues of G. Since the largest eigenvalue $\rho(G)$ of G is always main, we may order the eigenvalues of G so that the first $m(G)$ are main and all distinct eigenvalues of G are $\lambda_1, \ldots, \lambda_m, \ldots, \lambda_t$. It is well-known that a graph is regular if and only if it has exactly one main eigenvalue [2, p. 40].

In [6], Hoffman proved that G is a connected k-regular graph if and only if the polynomial $p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_t)$ satisfies

$$n \prod_{i=2}^{t} (A - \lambda_i I) = \prod_{i=2}^{t} (k - \lambda_i) \cdot J, \tag{1}$$

where I is the unit and J the all-one matrix. This result provides many applications for the investigation of graph structure by means of graph spectrum. In view of the importance of Hoffman identity in spectral graph theory, it is desired to generalize to non-regular graphs, recently, Dress and Steranović [4] derived a new version of Hoffman identity considering arbitrary expressions of the form $\prod_{i=1}^{m}(B - \beta_i I)$ for arbitrary matrix B, and deriving the generalizations of Hoffman identity to the harmonic and the semi-harmonic graphs.

Following [4], let B be a real symmetric matrix of order n, and a real numbers μ, let

$$U_\mu = \{\mathbf{f} \in R^n | B\mathbf{f} = \mu \mathbf{f}\},$$

and $\operatorname{spec}(B) = \{\mu \in R | \dim U_\mu > 0\}$ be the set of its all distinct eigenvalues.

If $\mathbf{f}_1, \ldots, \mathbf{f}_{n_\mu}$ is an orthonormal basis of U_μ for every $\mu \in \operatorname{spec}(B)$, and if $\{\beta_1, \beta_2, \ldots, \beta_t\} = \operatorname{spec}(B) \backslash \{\mu\}$ for some fixed real number $\mu \in R$ and $t = \#(\operatorname{spec}(B) \backslash \{\mu\})$, define Hoffman's matrix $H(B, \mu)$ by

$$H(B, \mu) = \prod_{\mu' \in \operatorname{spec}(B) \backslash \{\mu\}} (B - \mu' I)$$

then

$$H(B, \mu) = \prod_{\mu' \in \operatorname{spec}(B) \backslash \{\mu\}} (\mu - \mu') \cdot \sum_{p=1}^{n_\mu} \mathbf{f}_p \mathbf{f}_p^{\mathrm{T}}, \tag{2}$$

where $\mathbf{f}_1, \ldots, \mathbf{f}_{n_\mu}$ is any orthonormal basis of U_μ. In [4], Eq. (2) is called the Hoffman identity for matrix B. On the other hand, Hoffman identity for Laplace matrix of a graph was also presented in [8].

The aim of this note is to give the generalizations of Hoffman identities in [4,8] for more general cases. In Section 2 we present Hoffman-type identities for graphs by means of main eigenvalues. And in Section 3 we present Hoffman-type identities for arithmetical graphs [7].

2. Hoffman-type identity for a graph by main eigenvalues

For a graph G, the number of walks of length k of G starting at v is denoted by $d_k(v)$ and the number of all walks in G of length k by $W_k = W_k(G)$. Clearly, one has $d_0(v) = 1$, $d_1(v) = d(v)$, the degree of vertex v, $W_0 = n = |V|$, $W_1 = 2|E|$, and $d_{k+1}(v) = \sum_{u\sim v} d_k(u)$, for every $k \in N_0$. We call the matrix $W(G) = (\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{n-1}\mathbf{j})$ *walk matrix* of graph G (or matrix A), where $\mathbf{j}$ denotes the all one vector.

Recently, Hagos [5] showed that

Theorem 2.1 [5, Theorem 2.1]. *The rank of the walk-matrix $W(G)$ is equal to the number of main eigenvalues of G.*

Lemma 2.2 [5, Corollary 2.3]. *If G has exactly m main eigenvalues $\lambda_1, \ldots, \lambda_m$, then $\prod_{p\neq i}^{m}(A - \lambda_p I)\mathbf{j}$ is an eigenvector corresponding to λ_i, $i = 1, 2, \ldots, m$. In particular, $\prod_{p=1}^{m}(A - \lambda_p I)\mathbf{j} = 0$.*

Let all main eigenvalues of a graph G are $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $M(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)$. It was shown that $M(x)$ is a rational polynomial [1]. In fact, $M(x)$ is an integral polynomial by the following result (see [9] also).

Proposition 2.3. *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be all main eigenvalues of G. Then $M(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)$ is an integral polynomial.*

Proof. Since G has exactly m main eigenvalues, the rank of the walk-matrix $W(G)$ is m and $\mathbf{j}, A\mathbf{j}, \ldots, A^{m-1}\mathbf{j}$ are linear independent and $\mathbf{j}, A\mathbf{j}, \ldots, A^{m-1}\mathbf{j}, A^m\mathbf{j}$ are linear dependent. Thus there exist unique rational numbers $a_0, a_1, \ldots, a_{m-1}$ such that

$$A^m\mathbf{j} = a_0\mathbf{j} + a_1 A\mathbf{j} + \cdots + a_{m-1}A^{m-1}\mathbf{j}.$$

By Lemma 2.2, all a_i must be the elementary symmetric functions of $\lambda_1, \lambda_2, \ldots, \lambda_m$, $i = 0, 1, \ldots, m - 1$. Since $\lambda_1, \lambda_2, \ldots, \lambda_m$ are algebraic integers and the sums and products of algebraic integers are also algebraic integers, $a_0, a_1, \ldots, a_{m-1}$ are algebraic integers. Hence a_i are integers by the rationalities of a_i for $i = 0, 1, \ldots, m - 1$. □

By Eq. (2) and Lemma 2.2 we have Hoffman-type identity by means of main eigenvalues.

Theorem 2.4. *Let $\rho = \lambda_1, \ldots, \lambda_m$ be all main eigenvalues and $\lambda_1, \ldots, \lambda_m, \ldots, \lambda_t$ be all distinct eigenvalues of the connected graph G, and $\mathbf{f} = \prod_{i=2}^{m}(A - \lambda_i I)\cdot\mathbf{j}$. Then*

$$\langle \mathbf{f}, \mathbf{f}\rangle \prod_{i=2}^{t}(A - \lambda_i I) = \prod_{i=2}^{t}(\rho - \lambda_i)\cdot \mathbf{f}\mathbf{f}^{\mathrm{T}}.$$

A graph G is called to be 2-*walk* (a, b)-*linear* if there exist unique pair of rational numbers a, b (in fact, a, b are integers) such that

$$d_2(v) = ad(v) + b \tag{3}$$

holds for every vertex $v \in V(G)$. If $b = 0$ then 2-walk (a, b)-linear graph are called a-harmonic graphs in [3] and harmonic graphs have been investigated recently. Let G be an irregular connected graph. Hagos [5] showed that an irregular connected graph G has exactly two main eigenvalues if and only if G is 2-walk linear. Moreover, if G is an irregular 2-walk (a, b)-linear connected graph, then two main eigenvalues of G are $\lambda_{1,2} = \dfrac{a \pm \sqrt{a^2 + 4b}}{2}$.

Lemma 2.5. *Let G be an irregular graph with the largest eigenvalue λ_1. Then G has exactly two main eigenvalues λ_1 and λ_2 if and only if $(A - \lambda_2 I)\mathbf{j}$ is an eigenvector corresponding to λ_1 if and only if G is 2-walk $(\lambda_1 + \lambda_2, -\lambda_1\lambda_2)$-linear.*

Proof. If G has exactly two main eigenvalues λ_1 and λ_2, then $(A - \lambda_2 I)\mathbf{j}$ is an eigenvector corresponding to λ_1 follows Lemma 2.2. Conversely, if $(A - \lambda_2 I)\mathbf{j}$ is an eigenvector corresponding to λ_1, then $A(A - \lambda_2 I)\mathbf{j} = \lambda_1(A - \lambda_2 I)\mathbf{j}$. Thus

$$A^2\mathbf{j} = (\lambda_1 + \lambda_2)A\mathbf{j} - \lambda_1\lambda_2\mathbf{j},$$

and G has exactly two main eigenvalues λ_1, λ_2 follows from Theorem 2.1. $\square$

Theorem 2.6. *Let G be an irregular graph with the largest eigenvalues λ_1 and the least degree $\delta(G)$ more than λ_2. Then G is connected and with exactly two main eigenvalues λ_1, λ_2 if and only if G is connected 2-walk $(\lambda_1 + \lambda_2, -\lambda_1\lambda_2)$-linear if and only if*

$$\begin{aligned}(\lambda_1 - \lambda_2)(W_1 - \lambda_1 W_0)&\prod_{i=2}^{t}(A - \lambda_i I)\\ &= \prod_{i=2}^{t}(\lambda_1 - \lambda_i)\cdot(A - \lambda_2 I)\mathbf{j}[(A - \lambda_2 I)\mathbf{j}]^{\mathrm{T}}.\end{aligned} \tag{4}$$

Proof. Note that G is connected if and only if $\dim(U_{\lambda_1}) = 1$, and if G is a connected graph with exactly two main eigenvalues λ_1, λ_2 then $A^2\mathbf{j} = (\lambda_1 + \lambda_2)A\mathbf{j} - \lambda_1\lambda_2\mathbf{j}$, and

$$\begin{aligned}\langle (A - \lambda_2 I)\mathbf{j}, (A - \lambda_2 I)\mathbf{j}\rangle &= \mathbf{j}^{\mathrm{T}}(A - \lambda_2 I)(A - \lambda_2 I)\mathbf{j}\\ &= \mathbf{j}^{\mathrm{T}}(A^2\mathbf{j} - 2\lambda_2 A\mathbf{j} + \lambda_2^2\mathbf{j})\\ &= \mathbf{j}^{\mathrm{T}}[(\lambda_1 - \lambda_2)A + (\lambda_1^2 - \lambda_2\lambda_1)]\mathbf{j}\\ &= (\lambda_1 - \lambda_2)\mathbf{j}^{\mathrm{T}}(A - \lambda_1 I)\mathbf{j}\\ &= (\lambda_1 - \lambda_2)(W_1 - \lambda_1 W_0).\end{aligned}$$

Therefore, Eq. (4) holds from Theorem 2.4.

Conversely, if Eq. (4) holds, since $(A-\lambda_2 I)\mathbf{j}$ is a positive eigenvector, then G is connected and $\dfrac{(A-\lambda_2 I)\mathbf{j}}{(\lambda_1-\lambda_2)(W_2-\lambda_1 W_0)}$ is an unit eigenvector corresponding to λ_1, and G has exactly two main eigenvalues λ_1, λ_2 follows from Lemma 2.5. $\square$

Similar to Theorem 2.6, we have

Proposition 2.7. *Let $\rho = \lambda_1, \lambda_2, \lambda_3$ be all main eigenvalues and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_t$ be all distinct eigenvalues of a connected graph G, and $\mathbf{f} = (A-\lambda_2 I)(A-\lambda_3 I)\mathbf{j}$. Then*

$$c\prod_{i=2}^{t}(A-\lambda_i I) = \prod_{i=2}^{t}(\rho-\lambda_i)\cdot \mathbf{f}\mathbf{f}^{\mathrm{T}}, \tag{5}$$

where c is equal to $(\lambda_1^2 - 3\lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3)W_2 + \lambda_1(\lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_2^2 + \lambda_3^2)W_1 + \lambda_2^2\lambda_3^2 W_0 + \lambda_1\lambda_2\lambda_3(\lambda_1 - \lambda_2 - \lambda_3)$.

Remark 2.8. Let G be an irregular graph with the largest eigenvalues λ_1. Then G is harmonic if and only if G has exactly two main eigenvalues λ_1 and 0 and if G is strictly semi-harmonic then G has exactly three main eigenvalues $\lambda_1, 0, -\lambda_1$. Thus Eq. (4) and Eq. (5) generalize the results of [4] for harmonic and semi-harmonic graphs, respectively.

3. Hoffman-type identities for arithmetical graphs

Let $\mathbf{r} = (r_1, r_2, \ldots, r_n)^{\mathrm{T}}$ be a vector with $r_i \in Z_{\geqslant 1}$ and $\gcd(r_1, r_2, \ldots, r_n) = 1$ and we call $\mathbf{r}$ multiplicity vector.

An *arithmetical graph* $(G; M, \mathbf{r})$ consists of the following data:

(1) A connected graph G;
(2) a diagonal matrix $C = diag(c_1, c_2, \ldots, c_n)$ with $c_i \in Z_{\geqslant 1}$;
(3) an multiplicity vector $\mathbf{r}$ such that $M\mathbf{r} = 0$, where $M = C - A(G)$.

For an arithmetical graph $(G; M, \mathbf{r})$, we may say that $(M, \mathbf{r})$ define an arithmetical structure on G. Let $(G; M, \mathbf{r})$ be an arithmetical graph, the matrix M arise in algebraic geometry as intersection matrices of degenerating curves. On the other hand $(M, \mathbf{r})$ may be think as a generalization of Laplacian matrix of G since if $\mathbf{r} = \mathbf{j}$, M is the Laplacian matrix of G. Hoffman identity for Laplace matrix of a graph was presented in [8]. More about arithmetical graph see [7]. The following result is the generalization of Matrix-tree Theorem.

Theorem 3.1 [7, Proposition 1.1]. *Let* $(G; M, \mathbf{r})$ *be an arithmetical graph. Then matrix* M *satisfies the following properties*:

(1) M *is symmetric and positive semidefinite of rank* $n-1$ *and its kernel is generated over* Q *by* $\mathbf{r}$.

(2) *The adjoint of the matrix* M *is given by* $\operatorname{adj}(M) = \phi(M)\mathbf{r}\mathbf{r}^{\mathrm{T}}$, *where* $\phi(M)$ *is a positive integer*. (*In fact*, $\phi(M)$ *is the gcd of the determinants of all* $(n-1) \times (n-1)$ *minors of* M.)

For an arithmetical graph $(G; M, \mathbf{r})$, denote all distinct eigenvalues of M by $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_t > 0$ and the multiplicity of μ_i by $m(\mu_i)$. By Theorem 3.1, then $(-1)^{i+j} r_i r_j = \det M(i|j)$, where $M(i|j)$ is the $(n-1) \times (n-1)$ submatrix of M by removing the ith row and jth column. Hence

$$\mu_1^{m(\mu_1)} \mu_2^{m(\mu_2)} \cdots \mu_t^{m(\mu_t)} = (r_1^2 + r_2^2 + \cdots + r_t^2)\phi(M). \tag{6}$$

Notice that $\mathbf{r}$ is a basis of eigenspace of M associated eigenvalue 0, thus by Eq. (2) we have

Theorem 3.2. *For an arithmetical graph* $(G; M, \mathbf{r})$, *then*

$$\prod_{i=1}^{t} (M - \mu_i I) = (-1)^t h(M)\mathbf{r}\mathbf{r}^{\mathrm{T}}, \tag{7}$$

where $h(M) = \dfrac{\mu_1 \mu_2 \cdots \mu_t}{r_1^2 + r_2^2 + \cdots + r_n^2}$ *and call* $h(M)$ *the Hoffman number of arithmetical graph* $(G; M, \mathbf{r})$.

Since the matrix M of an arithmetical graph $(G; M, \mathbf{r})$ is an integer matrix, its eigenvalues are algebraic integers. Given an eigenvalue μ of M, let $p(x)$ be the monic irreducible polynomial over Q with integer coefficients such that $p(\mu) = 0$. The roots $\mu(1) = \mu, \mu(2), \ldots, \mu(s)$ of $p(x)$ are called *algebraic conjugates* of μ. Then the norm $N(\mu) = \mu(1)\mu(2)\cdots\mu(s)$ of $\mu \neq 0$ is a positive integer and $m(\mu(1)) = m(\mu(2)) = \cdots = m(\mu(s))$.

Corollary 3.3. *The Hoffman number* $h(M)$ *of an arithmetical graph is a positive integer*.

Proof. Note that if μ is an eigenvalue of M then $p(x)|\det(xI - M)$. Then all algebraic conjugates over Q of μ are also roots of the polynomial $(x-\mu_1)(x-\mu_2)\cdots(x-\mu_t)$. Hence by Galois theory, the polynomial $(x-\mu_1)(x-\mu_2)\cdots(x-\mu_t)$ has integer coefficients. Therefore the result follows from Theorem 3.2 and $\gcd(r_1, r_2, \ldots, r_n) = 1$. $\square$

By Eq. (6), we have

$$\tau(M) = \frac{\phi(M)}{h(M)} = \mu_1^{m_1-1}\mu_2^{m_2-1}\cdots\mu_t^{m_t-1}$$

is a rational algebraic integer and is therefore an integer. So we have

Proposition 3.4. *Let $(G; M, \mathbf{r})$ be an arithmetical graph. Then $h(M)|\phi(M)$.*

Proposition 3.5. *Let μ be a positive eigenvalue with multiplicity $m(\mu)$ of arithmetical graph $(G; M, \mathbf{r})$ with n vertices. Then $N(\mu)^{m(\mu)}|(r_1^2 + r_2^2 + \cdots + r_n^2)\phi(M)$ and $N(\mu)|(r_1^2 + r_2^2 + \cdots + r_n^2)h(M)$.*

Proof. By Eq. (6) and the definition of $h(M)$ then $\dfrac{(r_1^2 + r_2^2 + \cdots + r_n^2)\phi(M)}{N(\mu)^{m(\mu)}}$ and $\dfrac{(r_1^2 + r_2^2 + \cdots + r_n^2)h(M)}{N(\mu)}$ are rational algebraic integers and therefore integers. □

Proposition 3.6. *Let μ be a positive eigenvalue with multiplicity $m(\mu)$ of arithmetical graph $(G; M, \mathbf{r})$. Then $N(\mu)^{m(\mu)-1}|\phi(M)$. In particular, if $N(\mu)$ does not divide $\phi(M)$ then μ is simple.*

Proof. Since $\dfrac{\tau(M)}{N(\mu)^{m(\mu)-1}}$ is a rational algebraic integer and therefore an integer, and also $\tau(M)|\phi(M)$, we obtain the result. □

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